

The entanglement fidelity and quantum error correction

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Two new expressions for the *entanglement fidelity* recently introduced by Schumacher (LANL e-print quant-ph/9604023, to appear in Phys. Rev. A) are derived. These expressions show that it is the entanglement fidelity which must be maximized when performing error correction on qubits for quantum computers, not the fidelity, which is the most-often used generalization of the probability for storing a qubit correctly.

There is great interest in the use of entangled quantum states as a resource to accomplish tasks which are impossible or difficult to do classically. These tasks include quantum cryptography [1], quantum teleportation [2], quantum coding [3–5] and quantum computation [6,7]. For each of these tasks it is desirable to be able to maintain the entanglement of a subsystem with the remainder of the system.

A great deal of work has recently been done on quantum error correction; a sample of this work can be found in [8–14]. Much of this work has focused on preserving the state of a quantum system, without explicitly taking into account whether entanglement is preserved. Later in this Rapid Communication an example is given where the quantum state is preserved, while the entanglement is completely destroyed. First, though, we review what it means for the quantum state alone to be preserved, without accounting for entanglement.

Suppose, for example, that a single qubit is to be stored in the memory of a quantum computer. The qubit starts out in a pure state $|\psi\rangle$, and sometime later the interaction of the system with its environment has caused the state of the qubit to change to ρ . An obvious way to quantify how well the state has been stored is the quantity

$$P_e = 1 - \langle\psi|\rho|\psi\rangle, \quad (1)$$

which measures the probability of error in storage of the qubit. It would seem that this is the natural quantity to minimize in order to perform good quantum error correction. Equivalently, we wish to maximize the probability the state is stored correctly,

$$P_c = \langle\psi|\rho|\psi\rangle. \quad (2)$$

Clearly P_c is 1 if and only if the state is stored correctly, and less than 1 if some error has occurred during storage.

More generally we would like to have a measure of how successfully a mixed quantum state is stored. Suppose the initial quantum state is ρ_1 and the final quantum state is ρ_2 . What does it mean to say that these states are different, that is, can be distinguished? One measure of distinguishability with very useful properties is known as the *fidelity* [15] between the two states. It can be defined by the expression

$$F(\rho_1, \rho_2) := \max |\langle\psi_1|\psi_2\rangle|^2, \quad (3)$$

where the maximum is taken over all purifications [15], $|\psi_1\rangle$ and $|\psi_2\rangle$, of the states ρ_1 and ρ_2 .

One useful property of the fidelity is that it reduces to P_c in the case when ρ_1 is a pure state,

$$F(|\psi_1\rangle\langle\psi_1|, \rho_2) = \langle\psi_1|\rho_2|\psi_1\rangle. \quad (4)$$

The fidelity has many other desirable properties [15]. It is obviously symmetric in ρ_1 and ρ_2 . It satisfies the inequality

$$0 \leq F(\rho_1, \rho_2) \leq 1, \quad (5)$$

and $F(\rho_1, \rho_2) = 1$ if and only if $\rho_1 = \rho_2$. The smaller the fidelity, the more distinguishable the states are. These properties, together with the agreement with (2) in the pure state case make the fidelity an attractive measure of how well a quantum state has been preserved.

The next concept we introduce is that of an *extension* $\tilde{\rho}$ of ρ . Suppose ρ is defined on a system S which is part of a larger system ST which has density operator $\tilde{\rho}$, that is,

$$\text{tr}_T(\tilde{\rho}) = \rho, \quad (6)$$

where tr_T denotes the partial trace over T . Any density operator $\tilde{\rho}$ satisfying this condition will be called an extension of ρ . Notice that a purification $|\psi\rangle$ of ρ gives rise to an extension $|\psi\rangle\langle\psi|$ of ρ .

It follows easily from the definition of fidelity that if $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are extensions of ρ_1 and ρ_2 to the same larger system ST , then [15]

$$F(\tilde{\rho}_1, \tilde{\rho}_2) \leq F(\rho_1, \rho_2). \quad (7)$$

Physically what this means is that the density operators for the subsystem S are less distinguishable than those for the whole system ST , which is what we expect, since having access to the total system should not make it any more difficult to distinguish the states.

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One final useful property of fidelity concerns quantum operations. A quantum operation is the most general physically reasonable map that can be used to represent the change in a quantum state during interaction with an environment. Strictly speaking we are only concerned here with *non-selective* quantum operations in which the result of any measurement on the environment performed after the interaction is disregarded, that is, we are only concerned with the non-selective evolution of the quantum state. A detailed description of quantum operations can be found in [16] and a more physical account in the appendix to [17]. For our purposes we will need two (equivalent) ways of representing quantum operations. The first is the representation in terms of unitary operations with an ancilla. A quantum operation can always be represented by introducing an ancilla system E , and a unitary operator U on the system plus ancilla, SE , in such a way that the operation has the form

$$\mathcal{E}(\rho) = \text{tr}_E(U(\rho \otimes \sigma)U^\dagger), \quad (8)$$

where tr_E denotes tracing out the ancilla, and σ is the initial state of the ancilla. Conversely, any map of this form is a quantum operation.

The second representation for a quantum operation is the *operator sum* representation. In general any quantum operation can be written in the form

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger, \quad (9)$$

where the A_i are system operators satisfying the completeness relation $\sum_i A_i^\dagger A_i = I$. Conversely, any map of this form is a quantum operation.

How does the fidelity behave under quantum operations? It is not difficult to show [18] from (3), (7) and (8) that

$$F(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) \geq F(\rho_1, \rho_2). \quad (10)$$

That is, a quantum operation can only increase the fidelity between two states. This inequality is intuitively appealing since it says that physical processes can not increase the distinguishability of quantum states.

The fidelity is a very useful tool for analyzing the storage of states, but it does not take into account the possible entanglement of a system with other systems. A simple example to illustrate this point involves the storage of two two-level systems by Alice and Bob. In this example we will be considering Alice's system as the subsystem of interest, analogous to a single qubit of memory, while Bob is analogous to the remainder of the quantum computer.

Consider two possible dynamics for the combined system. The first is perfect storage of the entire system. If the initial density operator of the combined system is labelled ρ then this dynamics is given by the map

$$\rho \rightarrow \rho' = \rho. \quad (11)$$

Note that this map is clearly a quantum operation.

The second dynamics we will consider destroys ρ and leaves the system in a completely mixed state,

$$\rho \rightarrow \rho' = \frac{I \otimes I}{4}. \quad (12)$$

To see that this is a quantum operation we will use the representation (8) for quantum operations. Introduce an ancilla system consisting of two two-level systems ("Ted" and "Carol"). This ancilla is started in the state $\sigma = I \otimes I/4$. Suppose U is the unitary operator which swaps the state of system AB with the state of system TC . Then

$$\rho \rightarrow \text{tr}_{TC}(U(\rho \otimes \sigma)U^\dagger) = \frac{I \otimes I}{4} \quad (13)$$

is a quantum operation.

For Alice's system alone the corresponding dynamics are given by the quantum operations,

$$\rho_A \rightarrow \mathcal{E}_1(\rho_A) = \rho_A, \quad (14)$$

and

$$\rho_A \rightarrow \mathcal{E}_2(\rho_A) = \frac{I}{2}. \quad (15)$$

Suppose now that Alice and Bob start with a shared EPR pair,

$$|\psi\rangle = \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}. \quad (16)$$

The state of Alice's system alone is initially

$$\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|) = \frac{I}{2}, \quad (17)$$

where tr_B indicates a partial trace over Bob's system. Notice that under either dynamics the final state of Alice's system is given by

$$\rho'_A = \frac{I}{2}. \quad (18)$$

Thus under either dynamics Alice's system has been stored perfectly, that is, with fidelity equal to 1. However, the first dynamics leaves the entanglement of Alice's system with Bob's intact, while the second dynamics destroys the entanglement. Clearly, if we are interested in using entanglement as a resource, fidelity alone is not a sufficient measure of how well a quantum system is stored.

One could argue that what should be done is to look at the fidelity of the combined system belonging to Alice and Bob - this may be feasible in this simple example. However, in general, quantum computers can be very large systems compared to the subsystem (analogous to Alice's system) whose performance as a memory element we wish to analyze, and inclusion of the entire state and

dynamics of the quantum computer would make the analysis enormously complicated. What this example shows is that the fidelity of the subsystem density operators is not the correct quantity to look at to analyze the performance of the subsystem as a storage device if storing entanglement is important.

We will now define a quantity analogous to fidelity which does keep track of how well the state and entanglement of a subsystem of a larger system are stored, without requiring that the complete state or dynamics of the larger system be known. We will then prove that this quantity is equal to the *entanglement fidelity* defined by Schumacher [17]. Let us define a quantity, F_1 , by the expression

$$F_1(\rho, \mathcal{E}) := \min_{\tilde{\rho}, \mathcal{E}'} F\left((\mathcal{I}_S \otimes \mathcal{E}')(\tilde{\rho}), (\mathcal{E} \otimes \mathcal{E}')(\tilde{\rho})\right), \quad (19)$$

where the minimization is over all extensions $\tilde{\rho}$ of ρ to larger systems ST , and all possible quantum operations \mathcal{E}' that could occur on T . F_1 is a measure of how well the subsystem plus its entanglement with the remainder of the system is stored. We minimize over all possible extensions and dynamics for the remainder of the system in order to obtain the *worst possible value* the fidelity could have, regardless of the actual state or dynamics of the remainder of the system. Clearly to understand error correction for small parts of a quantum computer it would be desirable to use a quantity which depends only on the state of that part, not on the state of the entire computer, and the quantity F_1 is a natural candidate, since it measures the worst possible case.

A second quantity is also a useful measure of how well a system plus entanglement is stored. It will turn out that this quantity is equal to F_1 . Define

$$F_2(\rho, \mathcal{E}) := \min_{\tilde{\rho}} F\left(\tilde{\rho}, (\mathcal{E} \otimes \mathcal{I}_T)(\tilde{\rho})\right). \quad (20)$$

The motivation for this quantity is similar to that for F_1 , except now we assume that T is subject to the identity dynamics \mathcal{I}_T , instead of minimizing over all possible dynamics \mathcal{E}' for T . The main use of F_2 will be as an intermediate quantity.

We will now prove that F_1 and F_2 are equal. First, note that

$$F_1(\rho, \mathcal{E}) \leq F_2(\rho, \mathcal{E}), \quad (21)$$

since the minimization in F_1 clearly includes all the values being minimized over for F_2 . To see the reverse inequality, notice that

$$F\left(\tilde{\rho}, (\mathcal{E} \otimes \mathcal{I}_T)(\tilde{\rho})\right) \leq F\left((\mathcal{I}_S \otimes \mathcal{E}')(\tilde{\rho}), (\mathcal{E} \otimes \mathcal{E}')(\tilde{\rho})\right), \quad (22)$$

by (10), and thus

$$F_2(\rho, \mathcal{E}) \leq F_1(\rho, \mathcal{E}). \quad (23)$$

It follows that

$$F_1(\rho, \mathcal{E}) = F_2(\rho, \mathcal{E}). \quad (24)$$

Recently Schumacher [17] introduced a quantity called the *entanglement fidelity*, defined by the expression

$$F_e(\rho, \mathcal{E}) := \langle \psi | (\mathcal{E} \otimes \mathcal{I})(|\psi\rangle\langle\psi|) | \psi \rangle, \quad (25)$$

where $|\psi\rangle$ is any purification of ρ (Schumacher proves that any purification will give the same value), and $(\mathcal{E} \otimes \mathcal{I})$ is the natural extension of the evolution operator to the space on which ρ has been purified.

We will show that the entanglement fidelity is equal to the expressions F_1 and F_2 defined earlier. In particular we prove that

$$F_e(\rho, \mathcal{E}) = F_2(\rho, \mathcal{E}) = \min_{\tilde{\rho}} F\left(\tilde{\rho}, (\mathcal{E} \otimes \mathcal{I}_T)(\tilde{\rho})\right). \quad (26)$$

The proof is as follows. Write $F_e := F_e(\rho, \mathcal{E})$ and $F_2 := F_2(\rho, \mathcal{E})$. The minimization in F_2 includes states $\tilde{\rho} = |\psi\rangle\langle\psi|$ where $|\psi\rangle$ purifies ρ , and thus

$$\begin{aligned} F_2 &\leq F\left(|\psi\rangle\langle\psi|, (\mathcal{E} \otimes \mathcal{I})(|\psi\rangle\langle\psi|)\right) \\ &= \langle \psi | (\mathcal{E} \otimes \mathcal{I})(|\psi\rangle\langle\psi|) | \psi \rangle \\ &= F_e. \end{aligned} \quad (27)$$

To show the reverse inequality and thus complete the proof, suppose $\tilde{\rho}$ extends ρ to ST . Let $|\psi\rangle$ be a purification of $\tilde{\rho}$ on the space STU . It can be shown that $(\mathcal{E} \otimes \mathcal{I}_{TU})(|\psi\rangle\langle\psi|)$ is an extension of $(\mathcal{E} \otimes \mathcal{I}_T)(\tilde{\rho})$ by using an operator sum representation $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$, as follows,

$$\begin{aligned} &\text{tr}_U\left((\mathcal{E} \otimes \mathcal{I}_{TU})(|\psi\rangle\langle\psi|)\right) \\ &= \sum_i \text{tr}_U\left((A_i \otimes I_T \otimes I_U)|\psi\rangle\langle\psi|(A_i^\dagger \otimes I_T \otimes I_U)\right) \\ &= \sum_i (A_i \otimes I_T) \tilde{\rho} (A_i^\dagger \otimes I_T) \\ &= (\mathcal{E} \otimes \mathcal{I}_T)(\tilde{\rho}). \end{aligned} \quad (28)$$

Then from (7) we see that

$$\begin{aligned} F\left(\tilde{\rho}, (\mathcal{E} \otimes \mathcal{I}_T)(\tilde{\rho})\right) &\geq F\left(|\psi\rangle\langle\psi|, (\mathcal{E} \otimes \mathcal{I}_{TU})(|\psi\rangle\langle\psi|)\right) \\ &= \langle \psi | (\mathcal{E} \otimes \mathcal{I}_{TU})(|\psi\rangle\langle\psi|) | \psi \rangle. \end{aligned} \quad (29)$$

But since $|\psi\rangle$ is a purification of $\tilde{\rho}$ it is also a purification of ρ and thus

$$F_e = \langle \psi | (\mathcal{E} \otimes \mathcal{I}_{TU})(|\psi\rangle\langle\psi|) | \psi \rangle. \quad (30)$$

Combining the last two equations gives

$$F\left(\tilde{\rho}, (\mathcal{E} \otimes \mathcal{I})(\tilde{\rho})\right) \geq F_e. \quad (31)$$

Minimizing the left hand side of this inequality over all extensions $\tilde{\rho}$ of ρ tells us that $F_2 \geq F_e$. Combining this

with the inequality $F_2 \leq F_e$ found earlier, and the equality $F_1(\rho, \mathcal{E}) = F_2(\rho, \mathcal{E})$ gives the final result,

$$F_e(\rho, \mathcal{E}) = F_1(\rho, \mathcal{E}) = F_2(\rho, \mathcal{E}), \quad (32)$$

as required.

The expression (26) allows simple proofs of some of the properties of the entanglement fidelity. For example, we see immediately that

$$F_e(\rho, \mathcal{E}) \leq F(\rho, \mathcal{E}(\rho)), \quad (33)$$

since ρ is a trivial extension of itself, and thus is included in the minimization in (26). This is an intuitively reasonable result; it tells us that a state and its entanglement is not stored any better than the state alone. In the earlier example concerning Bob and Alice, the entanglement fidelity for Alice's system in the case of the first dynamics is $F_e(\rho_A, \mathcal{E}_1) = 1$, whereas for the second dynamics it is $F_e(\rho_A, \mathcal{E}_2) = \frac{1}{4}$, confirming our belief that the first dynamics preserves the state plus entanglement well, while the second dynamics preserves the state plus entanglement poorly. It should also be noted that for pure states,

$$F_e(|\psi\rangle\langle\psi|, \mathcal{E}) = \langle\psi|\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle \quad (34)$$

$$= F(|\psi\rangle\langle\psi|, \mathcal{E}(|\psi\rangle\langle\psi|)). \quad (35)$$

That the fidelity and entanglement fidelity are equal for pure states is intuitively reasonable, since pure states can not be entangled with other systems, and thus no entanglement can be destroyed during the storage process.

For completeness we will also mention the elegant explicit formula for entanglement fidelity derived in [17]. If the quantum operation \mathcal{E} is written in the form

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger, \quad (36)$$

then it can be shown that

$$F_e(\rho, \mathcal{E}) = \sum_i \text{tr}(A_i \rho) \text{tr}(A_i^\dagger \rho). \quad (37)$$

This form allows the entanglement fidelity to be calculated explicitly in actual examples.

A recent result due to Knill and Laflamme [10] gives another connection between the fidelity and entanglement fidelity. Theorem 5.3 of [10] shows that if

$$F(|\psi\rangle\langle\psi|, \mathcal{E}(|\psi\rangle\langle\psi|)) \geq 1 - \epsilon \quad (38)$$

for all pure states, $|\psi\rangle$, then

$$F_e(\rho, \mathcal{E}) \geq 1 - \frac{3\epsilon}{2}, \quad (39)$$

for all states ρ . That is, if the fidelity is kept high for all pure states, then it follows that the entanglement fidelity

is kept high for all states. This result shows that to preserve a quantum state and its entanglement accurately, it is sufficient to keep the fidelity of storage high, provided this is done for all pure states.

In this Rapid Communication two new expressions for the entanglement fidelity have been obtained. These show that it is the entanglement fidelity that is the important quantity to maximize in schemes for quantum error correction. It may also be useful in applications such as quantum teleportation, quantum cryptography and quantum coding in which entanglement may need to be preserved.

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